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AN ACCURATE METHOD FOR THE NUMERICAL SOLUTION OF FOURTH ORDER BOUNDARY VALUE PROBLEM BY GALERKIN METHOD WITH CUBIC B-SPLINES AS BASIS FUNCTIONS ALONG WITH EQUIDISTRIBUTION OF ERROR PRINCIPLE

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ABSTRACT

Fourth order two-point Boundary Value Problems (BVP) usually arise in various fields of science and engineering. In this article, cubic B-splines were utilised as basis functions for solving a fourth-order BVP by the Galerkin method and we have aimed to reduce the upper bound of the error of the numerical solutions obtained with the help of equidistribution of error principle (EDEP). Redefinition of basis functions was implemented to the initially chosen basis functions so that they vanish at all the Dirichlet boundary conditions. On applying the EDEP, the error is equidistributed in each sub-interval of the space variable domain. The proposed method was applied to several linear and non-linear BVPs. The non-linear BVPs were reduced to a sequence of linear BVPs using the concept of quasilinearization. The numerical results obtained are presented in the form of maximum absolute errors without and with applying EDEP, validating the proficiency and precision of the proposed method.

Keywords: cubic B-Splines, boundary value problem, Galerkin method, equidistribution of error principle, absolute error, quasilinearization.

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1. INTRODUCTION

Let us consider the most general form of fourth order BVP as follows:

$$q_0(t)u^{(4)}(t) + q_1(t)u^{(3)}(t) + q_2(t)u''(t) + q_3(t)u'(t) + q_4(t)u(t) = r(t), \quad z_1 < t < z_2 \tag{1}$$

subject to boundary conditions

$$u(z_1) = A_0, u(z_2) = C_0, u'(z_1) = A_1, u'(z_2) = C_1 \tag{2a}$$

or

$$u(z_1) = A_0, u(z_2) = C_0, \; u''(z_1) = A_2, \; u''(z_2) = C_2 \tag{2b}$$

or

$$u(z_1) = A_0, \; u(z_2) = C_0, \; u'(z_1) + \sigma_1 u(z_1) = A_3, u'(z_2) + \sigma_2 u(z_2) = C_3 \tag{2c}$$

where A_0, C_0, A_1, C_1, A_2, C_2, A_3, C_3, σ_1 and σ_2 are finite real constants and $q_0(t)$, $q_1(t)$, $q_2(t)$, $q_3(t)$, $q_4(t)$ and $r(t)$ are all continuous functions defined on the interval $[z_1, z_2]$.

Fourth order BVPs occur in numerous fields of science and engineering such as beam theory, fluid mechanics, diffusion reaction equations, biomechanics, model reaction behaviour of catalytic surfaces, electrochemical processes, etc.

An important tool assisting the pathway for solving these BVPs is the Splines. The spline curves were brought into action in shipbuilding mechanisms in the absence of computers and digital facilities while B-splines are used for modelling complex geometries in CAD and CAM. In 1966, the concept of B-splines was first introduced [1], though its numerical recurrence formula for computational purposes was discovered some years later [2, 3]. Cubic B-splines were utilised for solving the two-point BVP [4]. Solutions were also developed for a second order BVP using quadratic spline [5] while in progress to previous works, the quartic B-splines were used for solving the BVP of order three [6]. Third order BVP were also solved using non-polynomial splines [7]. On approximating a differential equation using a k^{th} order spline, the results obtained were accurate up to $(k+1)^{th}$ order [8].

The existence and uniqueness of the real valued function $u(t)$ were thoroughly studied which satisfies the BVP (1) with the given boundary conditions (2) [9]. In the convection-diffusion process, the asymptotic nature of the numerical procedures for solving the fourth order BVP was vividly discussed in [10] and the fourth order BVP based on the reaction-diffusion phenomenon was solved by the shooting method with higher accuracies [11]. The analytical solutions for such types of BVPs are available in very rare cases. Methods were developed for solving BVPs of order two and four using quintic and Sextic splines for studying the processes for plate deflection theory and other engineering applications [12, 13]. Also, numerical solutions for fourth order BVP were obtained using quartic splines [14]. Derivations of quintic spline and non-polynomial quintic spline methods were put to focus on obtaining a direct method and a numerical

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method for solving two-point BVP [15, 16]. In [17], certain variants of fourth order BVP were dealt with in the implementations of non-polynomial splines for obtaining a smooth approximation for the solutions.

Finite Element Method (FEM) is the most applied tool for solving the BVP due to the arbitrariness of the mesh element forms that we can work in complicated domains. Galerkin method is one of the variational methods in FEM, where the residual of the approximation is made orthogonal to the basis functions. In the light of using the Galerkin method, cubic B-splines were used to solve the fourth order BVP [18]. Solutions were also obtained for a general fourth order BVP using the Galerkin method with the help of Legendre polynomials [19]. For the higher order BVPs arising in fluid mechanics, the Galerkin method was implemented for better results [20]. In [21], the Sinc-Galerkin method was developed for solving the fourth order BVP using the double exponential (DE) transformations. In solving a fourth order BVP, the super convergence analysis was performed with the aid of using the locally discontinuous Galerkin method [22]. The equidistribution of Error Principle (EDEP) is a major technique used for reducing the error obtained by comparing the numerical and analytical results [23].

Considering the literature survey, the Galerkin method has been often used for solving BVPs since the Galerkin method provides a solution tending to the exact solution provided sufficient attention was paid to the boundary conditions. That means the basis functions should be zero at the boundary where the Dirichlet boundary conditions are mentioned. Here, the general linear fourth order BVP is solved using cubic B-splines as the basis functions. In this article, we have used various boundary conditions for the fourth order linear BVP, and the non-linear BVP was reduced to a sequence of linear BVPs using the quasilinearization technique [24]. Numerical solutions were obtained using the Galerkin method with basis functions as cubic B-splines and EDEP has been implemented to reduce the error obtained, which is the novelty of the present study.

2. DESCRIPTION OF THE PROPOSED METHOD

The numerical method used in this article comprises of Galerkin method as the variational technique along with cubic B-splines as the basis functions for solving a general linear fourth order BVP (1). The proposed method takes care of all possible boundary conditions (2a) - (2c) and generalizing it for solving any linear fourth order BVP.

2.1 Cubic B-Splines

In a closed interval $[z_1, z_2]$, we consider a cubic spline interpolate polynomial $s(t)$ (created using cubic B-splines) to a function with the knots chosen in such a way that they need not be evenly spaced.

$$z_1 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = z_2.$$

Introducing the six additional knots $t_{-3}, t_{-2}, t_{-1}, t_{n+1}, t_{n+2}$ and t_{n+3} such that

$$t_{-3} < t_{-2} < t_{-1} < t_0 \text{ and } t_n < t_{n+1} < t_{n+2} < t_{n+3}.$$

The spline functions $B_i(t)$ are defined as

$$B_i(t) = \begin{cases} \sum_{r=i-2}^{i+2} \frac{(t_r - t)_+^3}{\pi'(t_r)}, & if\ t \epsilon [t_{i-2}, t_{i+2}] \\ 0, & otherwise \end{cases}$$

where

$$(t_r - t)_+^3 = \begin{cases} (t_r - t)^3, & \text{if } t_r \geq t \\ 0, & \text{if } t_r \leq t \end{cases}$$

and $\pi(t) = (t - t_{i-2})(t - t_{i-1})(t - t_i)(t - t_{i+1})(t - t_{i+2})$.

The set $\{B_{-1}(t), B_0(t), \ldots, B_n(t), B_{n+1}(t)\}$ forms a basis for the space $S_3(t)$ of cubic polynomial splines. Among the family of non-zero splines, it was proven that in support of the knots $t_{-3} < t_{-2} < t_{-1} < t_0 < t_1 < \cdots < t_n < t_{n+1} < t_{n+2} < t_{n+3}$, cubic B-splines was having the smallest compact support [1].

We define the approximation for $u(t)$ as

$$u(t) = \sum_{j=-1}^{n+1} \alpha_j B_j(t) \tag{3}$$

where the nodal parameters $\alpha_j's$ are to be determined. At all the prescribed Dirichlet types of boundary conditions, the B-spline basis functions should vanish as per the definition of the Galerkin method. From the set of cubic B-splines $\{B_{-1}(t), B_0(t), B_1(t), B_2(t), \ldots, B_{n-1}(t), B_n(t), B_{n+1}(t)\}$, the basis functions $B_{-1}(t), B_0(t), B_1(t), B_{n-1}(t), B_n(t)$ and $B_{n+1}(t)$ do not vanish at one of the boundary points. Thus, a redefinition of the basis functions becomes necessary for them to vanish at all boundaries where Dirichlet boundary conditions are specified.

2.1.1 Redefinition of basis functions

On applying the Dirichlet boundary conditions mentioned in $(2a) - (2c)$ in Eq. (3), we get

$$u(z_1) = u(t_0) = \sum_{j=-1}^{1} \alpha_j B_j(t_0) = A_0 \tag{4}$$

$$u(z_2) = u(t_n) = \sum_{j=n-1}^{n+1} \alpha_j B_j(t_n) = C_0 \tag{5}$$

Get the expressions for the nodal parameters α_{-1}, α_{n+1} from equations (4) and (5), and apply them for the approximation of $u(t)$ given in equation (3), finally we get

$$u(t) = m(t) + \sum_{j=0}^{n} \alpha_j P_j(t) \tag{6}$$

Where

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$$m(t) = \frac{A_0}{B_{-1}(t_0)} B_{-1}(t) + \frac{C_0}{B_{n+1}(t_n)} B_{n+1}(t)$$

and

$$P_j(t) = \begin{cases} B_j(t) - \frac{B_j(t_0)}{B_{-1}(t_0)} B_{-1}(t), & \text{for } j = 0,1 \\ B_j(t), & \text{for } j = 2,3,\dots,n-2 \\ B_j(t) - \frac{B_j(t_n)}{B_{n+1}(t_n)} B_{n+1}(t), & \text{for } j = n-1, n \end{cases}$$

$\{P_j(t), j = 0,1,2,\dots,n-1,n\}$ is the newly obtained set of basis functions that have the property of vanishing at the boundaries where the Dirichlet type boundary conditions are prescribed and the Dirichlet boundary conditions defined in (2) are dealt by the function $m(t)$.

2.2 Application of the Galerkin Method to the Considered Problem

When we apply the Galerkin method to equation (1) with the newly obtained basis functions$P_i(t)$, we obtain

$$\int_{t_0}^{t_n} [q_0(t)u^{(4)}(t) + q_1(t)u^{(3)}(t) + q_2(t)u''(t) + q_3(t)u'(t) + q_4(t)u(t)]\, P_i(t)\, dt = \int_{t_0}^{t_n} r(t)P_i(t)\, dt \quad for\ i = 0,1,2,\dots,n-1,n. \tag{7}$$

2.2.1 Method with boundary condition (2a)

Applying integration by parts for the fourth and third order derivative terms of equation (7) followed by implementing the boundary condition$(2a)$, we obtain

$$\int_{t_0}^{t_n} q_0(t)u^{(4)}(t)P_i(t)\, dt = \left[-\frac{d}{dt}[q_0(t)P_i(t)]u''(t)\right]_{t_0}^{t_n} + \left[\frac{d^2}{dt^2}[q_0(t)P_i(t)]\right]_{t_n} C_1 - \left[\frac{d^2}{dt^2}[q_0(t)P_i(t)]\right]_{t_0} A_1 - \int_{t_0}^{t_n} \frac{d^3}{dt^3}[q_0(t)P_i(t)]\, u'(t)\, dt \tag{8}$$

$$\int_{t_0}^{t_n} q_1(t)u^{(3)}(t)P_i(t)\, dt = -\left[\frac{d}{dt}[q_1(t)P_i(t)]\right]_{t_n} C_1 + \left[\frac{d}{dt}[q_1(t)P_i(t)]\right]_{t_0} A_1 + \int_{t_0}^{t_n} \frac{d^2}{dt^2}[q_1(t)P_i(t)]\, u'(t)\, dt \tag{9}$$

On rearranging the terms in equation (7) after the substitution of equations(8), (9), and (6), the matrix form of the system of equations to get the nodal parameters is

$$K\alpha = L \tag{10}$$

where

$$K = [k_{ij}] = \int_{t_0}^{t_n} \Big[-\frac{d^3}{dt^3}[q_0(t)P_i(t)]P_j'(t) + \frac{d^2}{dt^2}[q_1(t)P_i(t)]P_j'(t) + q_2(t)P_i(t)P_j''(t) + q_3(t)P_i(t)P_j'(t) + q_4(t)P_i(t)P_j(t) \Big] dt - \left[\frac{d}{dt}[q_0(t)P_i(t)]P_j''(t)\right]_{t_0}^{t_n},$$

$$\text{for } i = 0,1,\dots,n\,, j = 0,1,\dots,n \tag{11}$$

$$L = [l_i] = \int_{t_0}^{t_n} \Big[r(t)P_i(t) + \frac{d^3}{dt^3}[q_0(t)P_i(t)]m'(t) - \frac{d^2}{dt^2}[q_1(t)P_i(t)]m'(t) - q_2(t)P_i(t)m''(t) - q_3(t)P_i(t)m'(t) - q_4(t)P_i(t)m(t) \Big] dt - \left[\frac{d^2}{dt^2}[q_0(t)P_i(t)]\right]_{t_n} C_1 + \left[\frac{d^2}{dt^2}[q_0(t)P_i(t)]\right]_{t_0} A_1 + \left[\frac{d}{dt}[q_1(t)P_i(t)]\right]_{t_n} C_1 - \left[\frac{d}{dt}[q_1(t)P_i(t)]\right]_{t_0} A_1 + \left[\frac{d}{dt}[q_0(t)P_i(t)]m''(t)\right]_{t_0}^{t_n},$$

$$for\ i = 0,1,\dots,n \tag{12}$$

and $\alpha = [\alpha_0\ \alpha_1\ \alpha_2 \dots\ \alpha_n]^T$.

2.2.2 Method with boundary condition (2b)

Applying integration by parts for the fourth and third order derivative terms of equation (7) followed by implementing the boundary condition$(2b)$, we obtain

$$\int_{t_0}^{t_n} q_0(t)u^{(4)}(t)P_i(t)\, dt = = -\left[\frac{d}{dt}[q_0(t)P_i(t)]\right]_{t_n} C_2 + \left[\frac{d}{dt}[q_0(t)P_i(t)]\right]_{t_0} A_2 + \int_{t_0}^{t_n} \frac{d^2}{dt^2}[q_0(t)P_i(t)]\, u''(t)\, dt \tag{13}$$

$$\int_{t_0}^{t_n} q_1(t)u^{(3)}(t)P_i(t)\, dt = -\int_{t_0}^{t_n} \frac{d}{dt}[q_1(t)P_i(t)]\, u''(t)\, dt \tag{14}$$

On rearranging the terms in equation (7) after the substitution of equations(13), (14), and (6), the matrix form of the system of equations to get the nodal parameters is

$$K\alpha = L \tag{15}$$

where

$$K = [k_{ij}] = \int_{t_0}^{t_n} \Big[\frac{d^2}{dt^2}[q_0(t)P_i(t)]P_j''(t) - \frac{d}{dt}[q_1(t)P_i(t)]P_j''(t) + q_2(t)P_i(t)P_j''(t) + q_3(t)P_i(t)P_j'(t) + q_4(t)P_i(t)P_j(t) \Big] dt$$

$$for\ i = 0,1,\dots,n\,, j = 0,1,\dots,n \tag{16}$$

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$$L = [l_i] = \int_{t_0}^{t_n} \Big[r(t)P_i(t) - \frac{d^2}{dt^2}[q_0(t)P_i(t)]m''(t) + \frac{d}{dt}[q_1(t)P_i(t)]m''(t) - q_2(t)P_i(t)m''(t) - q_3(t)P_i(t)m'(t) - q_4(t)P_i(t)m(t) \Big] dt + \left[\frac{d}{dt}[q_0(t)P_i(t)]\right]_{t_n} C_2 - \left[\frac{d}{dt}[q_0(t)P_i(t)]\right]_{t_0} A_2, \text{ for } i = 0,1,\dots,n \tag{17}$$

and $\alpha = [\alpha_0\ \alpha_1\ \alpha_2 \dots\ \alpha_n]^T$.

2.2.3 Method with boundary condition (2c)

Applying integration by parts for the fourth and third order derivative terms of equation (7) followed by implementing the boundary condition$(2c)$, we obtain

$$\int_{t_0}^{t_n} q_0(t)u^{(4)}(t)P_i(t)\,dt = \left[-\frac{d}{dt}[q_0(t)P_i(t)]u''(t)\right]_{x_0}^{x_n} + \left[\frac{d^2}{dt^2}[q_0(t)P_i(t)]\right]_{t_n} (C_3 - \sigma_2 C_0) - \left[\frac{d^2}{dt^2}[q_0(t)P_i(t)]\right]_{t_0} (A_3 - \sigma_1 A_0) - \int_{t_0}^{t_n} \frac{d^3}{dt^3}[q_0(t)P_i(t)]\,u'(t)\,dt \tag{18}$$

$$\int_{t_0}^{t_n} q_1(t)u^{(3)}(t)P_i(t)\,dt = -\left[\frac{d}{dt}[q_1(t)P_i(t)]\right]_{t_n} (C_3 - \sigma_2 C_0) + \left[\frac{d}{dt}[q_1(t)P_i(t)]\right]_{t_0} (A_3 - \sigma_1 A_0) + \int_{t_0}^{t_n} \frac{d^2}{dt^2}[q_1(t)P_i(t)]\,u'(t)\,dt \tag{19}$$

On rearranging the terms in equation (7) after the substitution of equations(18), (19), and (6), the matrix form of the system of equations to get the nodal parameters is

$$K\alpha = L \tag{20}$$

where

$$K = [k_{ij}] = \int_{t_0}^{t_n} \Big[-\frac{d^3}{dt^3}[q_0(t)P_i(t)]P_j'(t) + \frac{d^2}{dt^2}[q_1(t)P_i(t)]P_j'(t) + q_2(t)P_i(t)P_j''(t) + q_3(t)P_i(t)P_j'(t) + q_4(t)P_i(t)P_j(t) \Big] dt - \left[\frac{d}{dt}[q_0(t)P_i(t)]P_j''(t)\right]_{t_0}^{t_n}, \ for\ i = 0,1,\dots,n, j = 0,1,\dots,n \tag{21}$$

$$L = [l_i] = \int_{t_0}^{t_n} \Big[r(t)P_i(t) + \frac{d^3}{dt^3}[q_0(t)P_i(t)]m'(t) - \frac{d^2}{dt^2}[q_1(t)P_i(t)]m'(t) - q_2(t)P_i(t)m''(t) - q_3(t)P_i(t)m'(t) - q_4(t)P_i(t)m(t) \Big] dt - \left[\frac{d^2}{dt^2}[q_0(t)P_i(t)]\right]_{t_n} (C_3 - \sigma_2 C_0) + \left[\frac{d^2}{dt^2}[q_0(t)P_i(t)]\right]_{t_0} (A_3 - \sigma_1 A_0) + \left[\frac{d}{dt}[q_1(t)P_i(t)]\right]_{t_n} (C_3 - \sigma_2 C_0) - \left[\frac{d}{dt}[q_1(t)P_i(t)]\right]_{t_0} (A_3 - \sigma_1 A_0) + \left[\frac{d}{dt}[q_0(t)P_i(t)]m''(t)\right]_{t_0}^{t_n}, \ for\ i = 0,1,\dots,n \tag{22}$$

and $\alpha = [\alpha_0\ \alpha_1\ \alpha_2 \dots\ \alpha_n]^T$.

3. PROCEDURE OF SOLVING THE NODAL PARAMETERS

In the stiff matrix K, the general integral element is

$$\sum_{m=0}^{n-1} I_m$$

where $I_m = \int_{t_m}^{t_{m+1}} \Xi_i(t)\Xi_j(t)H(t)\,dt$ and $\Xi_i(t)$, $\Xi_j(t)$ are the cubic B-splines or their derivatives. If $(t_{i-2}, t_{i+2}) \cap (t_{j-2}, t_{j+2}) \cap (t_m, t_{m+1}) = \emptyset$, then we can conclude that $I_m = 0$. The 4-point Gauss-Legendre quadrature formula has been implemented for evaluating each I_m and thus we obtain a seven diagonal band matrix K. The band matrix solution package was utilized for solving $K\alpha = L$ in order to obtain the nodal parameter vector α. To solve the BVP $(1) - (2)$, we have used FORTRAN-90 code.

4. EQUIDISTRIBUTION OF ERROR PRINCIPLE (EDEP)

According to EDEP, we consider a space variable domain where we equidistribute the obtained errors in each interval of the given domain. Due to this, the error of approximation is less when compared with the usual methods.

Assuming that we are using a k^{th} order piecewise polynomial Ω for approximating a n^{th} order differential equation of ω with the independent variable t lying in $[z_1, z_2]$, we divide $[z_1, z_2]$ into n sub-intervals such that

$$z_1 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = z_2$$

Suppose that

$h_i = t_i - t_{i-1}$, for $i = 1,2,\dots,n$

and

$h = \max_i(h_i)$

Under the presumption of h attaining very small values, the approximation error is defined as [23]

$$\|\omega - \Omega\|_\infty \le constant \max_i |h_i|^k \left\|\frac{d^k\omega}{dt^k}\right\|_{(i)}$$

Here, $\left\|\frac{d^k\omega}{dt^k}\right\|_{(i)}$ is representing the sup-norm of the k^{th} derivative of ω concerning t in the i^{th} interval.

The points $t_1, t_2, \dots, t_{n-1}$ are the interior points, which are placed for minimizing $\max_i |h_i|^k \left\|\frac{d^k\omega}{dt^k}\right\|_{(i)}$. This is achieved when the above points are placed in such a way that $|h_i|^k \left\|\frac{d^k\omega}{dt^k}\right\|_{(i)} = constant,\ for\ i = 1,2,\dots,n$.

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Since $\frac{d^k\omega}{dt^k}$ is unknown to us, determining the exact values of $t_1, t_2, \dots, t_{n-1}$ becomes quite difficult.

Since Ω approximates ω, we can approximate $\frac{d^k\omega}{dt^k}$ by $\frac{d^k\Omega}{dt^k}$. Therefore, the placement of the nodes $t_1, t_2, \dots, t_{n-1}$ can be done by taking

$$\int_{t_i}^{t_{i+1}} \left\|\frac{\partial^k\Omega}{\partial t^k}\right\|^{1/k} dt = \frac{1}{n}\int_{z_1}^{z_2} \left\|\frac{\partial^k\Omega}{\partial t^k}\right\|^{1/k} dt \quad for\ i = 0,1,2,\dots,n-1.$$

The $t_1, t_2, \dots, t_{n-1}$ values can be found by solving the above equation by using the Newton-Raphson method.

5. RESULTS AND DISCUSSIONS

We have considered various forms of fourth order BVP, and the maximum absolute errors have been computed accordingly in order to test the applicability of the proposed method. Initially, mesh points are chosen in such a way that we have equal subintervals with step length $h = (z_1 - z_2)/n$, where $[z_1, z_2]$ is the space variable (working) domain and this domain is divided into n sub-intervals. On applying EDEP, the existing mesh points were re-distributed to reduce the maximum absolute error in each working subinterval of the domain.

Example 1. Consider the following BVP

$$z^{(4)} + 4z = 1, -1 < t < 1$$

along with the prescribed boundary conditions

$$z(-1) = 0, z(1) = 0, z'(-1) = \frac{\mathrm{Sinh}(2) - \mathrm{Sin}(2)}{4(\mathrm{Cosh}(2) + \mathrm{Cos}(2))}, z'(1) = \frac{\mathrm{Sin}(2) - \mathrm{Sinh}(2)}{4(\mathrm{Cosh}(2) + \mathrm{Cos}(2))}.$$

The exact solution of the BVP is

$$z(t) = 0.25\left[1 - 2\left(\frac{\mathrm{Sinh}(1)\mathrm{Sin}(1)\mathrm{Sinh}(x)\,\mathrm{Sin}(x) + \mathrm{Cosh}(1)\mathrm{Cos}(1)\mathrm{Cosh}(x)\mathrm{Cos}(x)}{\mathrm{Cos}(2) + \mathrm{Cosh}(2)}\right)\right].$$

For solving the above linear BVP using our method, we have used 10 subintervals for dividing the working domain$[-1,1]$. The obtained maximum absolute errors for this problem without and with applying EDEP are given in Table-1.

Table-1. Maximum absolute error comparison for pre and post application of EDEP.

	Before applying EDEP	After applying EDEP
Maximum Absolute Error	9.271502×10^{-5}	6.490052×10^{-7}

Example 2. Consider the following BVP

$$z^{(4)} + tz = -(8 + 7t + t^3)e^t, \quad 0 < t < 1$$

along with the prescribed boundary conditions

$$z(0) = 0, z(1) = 0, z'(0) = 1, z'(1) = -e.$$

The exact solution of the BVP is

$$z(t) = t(1-t)e^t.$$

For solving the above linear BVP using our method, we have used 10 subintervals for dividing the working domain[0,1]. The obtained maximum absolute errors for this problem without and with applying EDEP are given in Table-2.

Table-2. Maximum absolute error comparison for pre and post application of EDEP.

	Before applying EDEP	After applying EDEP
Maximum Absolute Error	5.131960×10^{-5}	4.978001×10^{-7}

Example 3. Consider the following BVP

$$z^{(4)} - 3601z'' + 3600z = -1 + 1800t^2, 0 < t < 1$$

along with the prescribed boundary conditions

$$z(0) = 1, z(1) = 1.5 + \mathrm{Sinh}(1), z'(0) - z(0) = 0, z'(1) - z(1) = -0.5 + \mathrm{Cosh}(1) - \mathrm{Sinh}(1).$$

The exact solution of the BVP is

$$z(t) = 1 + 0.5\,t^2 + Sinh(t).$$

For solving the above linear BVP using our method, we have used 10 subintervals for dividing the working domain[0,1]. The obtained maximum absolute errors for this problem without and with applying EDEP are given in Table-3.

Table-3. Maximum absolute error comparison for pre and post application of EDEP.

	Before applying EDEP	After applying EDEP
Maximum Absolute Error	9.775162×10^{-6}	3.871918×10^{-7}

Example 4. Consider the following BVP

$$z^{(4)} - z = -4\big(2tCos(t) + 3Sin(t)\big), \quad 0 < t < 1$$

along with the prescribed boundary conditions

$$z(0) = 0, z(1) = 0, z''(0) = 0, z''(1) = 4\mathrm{Cos}(1) + 2\mathrm{Sin}(1).$$

The exact solution of the BVP is

$$z(t) = (t^2 - 1)Sin(t).$$

For solving the above linear BVP using our method, we have used 10 subintervals for dividing the working domain[0,1]. The obtained maximum absolute errors for this problem without and with applying EDEP are given in Table-4.

Table-4. Maximum absolute error comparison for pre and post application of EDEP.

	Before applying EDEP	After applying EDEP
Maximum Absolute Error	1.120567×10^{-5}	8.068085×10^{-8}

Example 5. Consider the following non-linear BVP

$$z^{(4)} = z^2 - t^{10} + 4t^9 - 4t^8 - 4t^7 + 8t^6 - 4t^4 + 120t - 48, 0 < t < 1$$

along with the prescribed boundary conditions

$$z(0) = 0, z(1) = 1, z'(0) = 0, z'(1) = 1.$$

The exact solution of the BVP is

$$z(t) = t^5 - 2t^4 + 2t^2.$$

To solve the above nonlinear BVP, firstly it is converted into a sequence of linear BVPs by using the quasilinearization technique [24] as

$$z^{(4)}_{(n+1)} - \left[2z_{(n)}\right]z_{(n+1)} = -t^{10} + 4t^9 - 4t^8 - 4t^7 + 8t^6 - 4t^4 + 120t - 48 - \left[z_{(n)}\right]^2, n = 0,1,2,\ldots$$

along with the boundary conditions

$$z_{(n+1)}(0) = 0, z_{(n+1)}(1) = 1, z'_{(n+1)}(0) = 0, z'_{(n+1)}(1) = 1.$$

Here $z_{(n+1)}$ is the $(n+1)^{th}$ approximation for z. For solving the above sequence of linear BVPs using our method, we have used 10 subintervals for dividing the working domain[0,1]. The obtained maximum absolute errors for this problem without and with applying EDEP are given in Table-5.

Table-5. Maximum absolute error comparison for pre and post application of EDEP.

	Before applying EDEP	After applying EDEP
Maximum Absolute Error	6.872416×10^{-5}	1.855552×10^{-7}

Example 6. Consider the following non-linear BVP

$$z^{(4)} = Sin(t) + Sin^2(t) - [z'']^2, \ 0 < t < 1$$

along with the prescribed boundary conditions

$$z(0) = 0, z(1) = Sin(1), z'(0) = 1, z'(1) = Cos(1).$$

The exact solution of the BVP is

$$z(t) = Sin(t).$$

To solve the above nonlinear BVP, firstly it is converted into a sequence of linear BVPs by using the quasilinearization technique [24] as

$$z^{(4)}_{(n+1)} + \left[2z''_{(n)}\right]z''_{(n+1)} = Sin(t) + Sin^2(t) + \left[z''_{(n)}\right]^2, n = 0,1,2,\ldots$$

along with the boundary conditions

$$z_{(n+1)}(0) = 0, z_{(n+1)}(1) = Sin(1), z'_{(n+1)}(0) = 1, z'_{(n+1)}(1) = Cos(1).$$

Here $z_{(n+1)}$ is the $(n+1)^{th}$ approximation for z. For solving the above sequence of linear BVPs using our method, we have used 10 subintervals for dividing the working domain[0,1]. The obtained maximum absolute errors for this problem without and with applying EDEP are given in Table-6.

Table-6. Maximum absolute error comparison for pre and post application of EDEP.

	Before applying EDEP	After applying EDEP
Maximum Absolute Error	7.218122×10^{-5}	7.073760×10^{-7}

Example 7. Consider the following non-linear BVP

$$z^{(4)} - 6e^{-4z} = -\frac{12}{(1+t)^4}, 0 < t < 1$$

along with the prescribed boundary conditions

$$z(0) = 0, z(1) = ln(2), z'(0) = 1, z'(1) = 0.5.$$

The exact solution of the BVP is
$z(t) = ln(1+t)$.

To solve the above nonlinear BVP, firstly it is converted into a sequence of linear BVPs by using the quasilinearization technique [24]:

$$z_{(n+1)}^{(4)} + [24e^{-4z_{(n)}}]z_{(n+1)} = -\frac{12}{(1+t)^4} + e^{-4z_{(n)}}[6 + 24z_{(n)}], \quad n = 0,1,2,\ldots$$

along with the boundary conditions

$$z_{(n+1)}(0) = 0, z_{(n+1)}(1) = ln(2), z'_{(n+1)}(0) = 1, z'_{(n+1)}(1) = 0.5.$$

Here $z_{(n+1)}$ is the $(n+1)^{th}$ approximation for z. For solving the above sequence of linear BVPs using our method, we have used 10 subintervals for dividing the working domain[0,1]. The obtained maximum absolute errors for this problem without and with applying EDEP are given in Table-7.

Table-7. Maximum absolute error comparison for pre and post application of EDEP.

	Before applying EDEP	After applying EDEP
Maximum Absolute Error	6.124377×10^{-5}	3.552139×10^{-7}

Example 8. Consider the following non-linear BVP

$$z^{(4)} + \frac{t^2}{1+z^2} = -72(1-5t+5t^2) + \frac{t^2}{1+(t-t^2)^6}, \quad 0 < t < 1$$

along with the prescribed boundary conditions

$$z(0) = 0, z(1) = 0, z'(0) = 0, z'(1) = 0.$$

The exact solution of the BVP is

$$z(t) = t^3(1-t)^3.$$

To solve the above nonlinear BVP, firstly it is converted into a sequence of linear BVPs by using the quasilinearization technique [24] as

$$z_{(n+1)}^{(4)} - \frac{2t^2 z_{(n)}}{\left(1+[z_{(n)}]^2\right)^2} z_{(n+1)} = -72(1-5t+5t^2) + \frac{t^2}{1+(t-t^2)^6} - \frac{2t^2[z_{(n)}]^2}{\left(1+[z_{(n)}]^2\right)^2} - \frac{t^2}{1+[z_{(n)}]^2}, \quad n = 0,1,2,\ldots$$

along with the boundary conditions

$$z_{(n+1)}(0) = 0, z_{(n+1)}(1) = 0, z'_{(n+1)}(0) = 0, z'_{(n+1)}(1) = 0.$$

Here $z_{(n+1)}$ is the $(n+1)^{th}$ approximation for z. For solving the above sequence of linear BVPs using our method, we have used 10 subintervals for dividing the working domain[0,1]. The obtained maximum absolute errors for this problem without and with applying EDEP are given in Table-8.

Table-8. Maximum absolute error comparison for pre and post application of EDEP.

	Before applying EDEP	After applying EDEP
Maximum Absolute Error	9.820797×10^{-7}	4.774332×10^{-9}

Example 9. Consider the following non-linear BVP

$$z^{(4)} + z^2 = -8t\text{Cos}(t) - 13\text{Sin}(t) + t^2\text{Sin}(t) + (t^2-1)^2\text{Sin}^2(t), 0 < t < 1$$

along with the prescribed boundary conditions

$$z(0) = 0, z(1) = 0, z''(0) = 0, z''(1) = 2\text{Sin}(1) + 4\text{Cos}(1).$$

The exact solution of the BVP is

$$z(t) = (t^2-1)\text{Sin}(t).$$

To solve the above nonlinear BVP, firstly it is converted into a sequence of linear BVPs by using the quasilinearization technique [24] as

$$z_{(n+1)}^{(4)} + [2z_{(n)}]z_{(n+1)} = [z_{(n)}]^2 - 8t\text{Cos}(t) - 13\text{Sin}(t) + t^2\text{Sin}(t) + (t^2-1)^2\text{Sin}^2(t), \quad n = 0,1,2,\ldots$$

along with the boundary conditions

$$z_{(n+1)}(0) = 0, z_{(n+1)}(1) = 0, z''_{(n+1)}(0) = 0, z''_{(n+1)}(1) = 2\text{Sin}(1) + 4\text{Cos}(1).$$

Here $z_{(n+1)}$ is the $(n+1)^{th}$ approximation for z. For solving the above sequence of linear BVPs using our method, we have used 10 subintervals for dividing the working domain[0,1]. The obtained maximum absolute errors for this problem without and with applying EDEP are given in Table-9.

Table-9. Maximum absolute error comparison for pre and post application of EDEP.

	Before applying EDEP	After applying EDEP
Maximum Absolute Error	9.506941×10^{-6}	6.654859×10^{-8}

CONCLUSIONS

The motivation for conducting this study lies in numerically solving fourth order BVPs arising in numerous fields of science and engineering such as beam theory, fluid mechanics, diffusion reaction equations, biomechanics, model reaction behaviour of catalytic surfaces, electrochemical process, etc. In this article, we have developed a numerical method to solve a fourth order BVP using the Galerkin method with cubic B-splines as basis functions. The Cubic B-splines are redefined into a new set of basis functions that vanish on the boundary where the Dirichlet boundary conditions are specified. The upper bound of the error was reduced with the help of EDEP. The accuracy and strength of the proposed method were tested based on the application of our method on four linear and five non-linear fourth order BVPs with various boundary conditions, and they were found to be quite satisfactory. The reduction in the maximum absolute error values for the concerned problems using EDEP provides a strong foundation for our method.

DECLARATIONS

Conflict of interest: The authors have no relevant financial or non-financial interest to disclose.

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